

On Some Axioms for Ranking Sets of Alternatives*

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The problem of extending an ordering on a finite set of alternatives to its power set is considered. It is shown that two fairly mild axioms imply the restrictive condition that every set is equivalent to the set consisting only of its least and greatest elements. A characterisation of all extensions of a linear ordering, which satisfy our two axioms, is made by means of a class of real valued functions defined on integer pairs. The induced orderings are interpreted in terms of choice under uncertainty and an application made to welfare economics. *Journal of Economic Literature* Classification Numbers: 025, 026.

I. INTRODUCTION

Let X be a given set of alternatives with χ denoting $2^X - \{\emptyset\}$, and let R be a given linear ordering over X . A number of writers have recently considered the problem of inducing an ordering \succcurlyeq over χ given the ordering over X .¹

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¹ Among others see Kannai and Peleg [8], Fishburn [4], Packard [12], Heiner and Packard [7], and Barbera and Pattanaik [2].

The purpose of this paper is to explore the implications of certain axioms regarding the relationship between \succsim and R . The formal problem of inducing an ordering \succsim over χ given the ordering R over X admits many possible interpretations (for a discussion of some of these interpretations see Packard [12], Kannai and Peleg [8], Barbera and Pattanaik [3], and Heiner and Packard [7]). However, the specific interpretation which we adopt in this paper for the purpose of illustration runs in terms of an agent's choice under uncertainty (when probabilities, subjective or objective, are absent). Each element A in χ is visualized as the set of all possible consequences of some action of the agent. It is assumed that the agent does not have any probabilities associated with the alternatives in A . Thus the problem of choosing an action is one of uncertainty rather than one of risk, to use the terminology of Luce and Raiffa [11]. Under this interpretation, the ranking of the elements of χ is to be intuitively interpreted as reflecting the relative desirability of the actions associated with these different elements of χ .

Kannai and Peleg [8] and Barbera and Pattanaik [3] have shown the inconsistency of certain plausible axioms regarding the relationship between the ordering \succsim over χ and the linear ordering R over X . Given this background of negative results we have chosen two of the weakest axioms discussed in this literature. It is shown that while these plausible axioms do not lead to impossibility results, they nevertheless impose severe restrictions on the ordering \succsim . The restriction takes the form of rendering every nonempty subset A of X indifferent (in terms of \succsim) to the set of R -greatest and R -least elements in A . (This is similar to the conclusion of Arrow and Hurwicz [1], who used a different framework involving explicitly states of the world and an entirely different set of axioms.) Thus an agent acting under uncertainty, who satisfies our axioms, will rank alternative actions exclusively on the basis of the best and worst results yielded by these actions. Following this result we characterize all orderings over χ which satisfy our two axioms given a linear ordering R over X . We conclude with an application of our results to a problem in welfare economics. A distinguished tradition in welfare economics² identifies a person's ranking in terms of social welfare, of alternative social states, as being given by the rational egoistic preferences of the same individual over certain types of uncertain prospects. In this framework, a reinterpretation of our formal results implies that for an individual whose egoistic behaviour under uncertainty satisfies our axioms, the social welfare ranking of any two social states will be based exclusively on the welfare of the best-off and the worst-off individuals in those two social states.

² See Rawls [13] and Harsanyi [6].

II. THE NOTATION AND DEFINITIONS

As specified in the Introduction, X , χ , and R will respectively indicate a given set of alternatives (assumed to be finite), the set of all nonempty subsets of X , and a given linear ordering over X . \succcurlyeq indicates a binary relation ("at least as good as") over χ . The asymmetric factor of R is indicated by P . The asymmetric and symmetric factors of \succcurlyeq are indicated by $>$ and \sim , respectively.

For convenience, we index the alternatives in X as x_1, \dots, x_n , in such a way that $(x_n P x_{n-1} \& x_{n-1} P x_{n-2} \& \dots \& x_2 P x_1)$. For all $A \in \chi$, N_A stands for $\{i \in \{1, \dots, n\} \mid x_i \in A\}$, and $g(A)$ and $\ell(A)$, respectively, stand for the R -greatest element and the R -least element in A .

DEFINITION 2.1. With respect to the given ordering R over X , a binary relation \succcurlyeq over χ satisfies

(2.1.1) *the Weak Dominance Principle (WDP)* iff for all $x, y \in X$, xPy implies $\{\{x\} > \{x, y\} \& \{x, y\} > \{y\}\}$;

(2.1.2) *Positive Responsiveness (PR)* iff for all $A \in \chi$ and for all $x \in X$, $[(xPy \text{ for all } y \in A) \text{ implies } A \cup \{x\} > A]$ and $[(yPx \text{ for all } y \in A) \text{ implies } A > A \cup \{x\}]$;

(2.1.3) *Independence (IND)* iff for all $A, B, C \in \chi$, $[A > B \text{ and } A \cap C = B \cap C = \emptyset]$ implies $[A \cup C > B \cup C]$.

(2.1.4) *Weak Independence (WIND)* iff for all $A, B, C \in \chi$, $[A > B \text{ and } A \cap C = B \cap C = \emptyset]$ implies $[A \cup C \succcurlyeq B \cup C]$.

Conditions (1) WDP, (2) PR, (3) IND, and (4) WIND have been respectively discussed among others by (1) Barbera [2] and Barbera and Pattanaik [3]; (2) Gardenfors [5] and Kannai and Peleg [8]; (3) Krantz, Luce, Suppes, and Tversky [10], Packard [12], and Barbera and Pattanaik [3]; and (4) Kannai and Peleg [8].

WDP is an intuitively transparent condition. For an intuitive discussion of IND and WIND, the reader may refer to Barbera and Pattanaik [3]. It is obvious that WDP is weaker than PR, and WIND is weaker than IND.

III. THE RESULTS

First, we note the following negative results due to Kannai and Peleg [8] and Barbera and Pattanaik [3].

THEOREM 3.1 (Kannai and Peleg). *If $|X| \geq 6$, then there does not exist any ordering over χ which satisfies PR and WIND with respect to R .*

THEOREM 3.2 (Barbera and Pattanaik). *If $|X| \geq 3$, then there does not exist any binary relation over χ satisfying WDP and IND with respect to R .*

Given these negative results we investigate here the implications of imposing, on an ordering \geq over χ , properties WDP (which is weaker than PR used by Kannai and Peleg) and WIND (which is weaker than IND used by Barbera and Pattanaik).

THEOREM 3.3. *Let \geq be an ordering over χ satisfying WDP and WIND with respect to R . Then for all $A \in \chi$, $A \sim \{g(A), \ell(A)\}$.*

Proof. We first show that

$$\text{for all } D \subseteq A \text{ such that } g(A), \ell(A) \in D, \text{ and for all } x \in D - \{g(A), \ell(A)\}, D - \{x\} \sim D. \quad (3.1)$$

Let $D \subseteq A$, $g(A), \ell(A) \in D$; and $x \in D - \{g(A), \ell(A)\}$. Then $g(A) P x$ and $x P \ell(A)$.

Hence by WDP,

$$\{g(A)\} > \{g(A), x\} \quad (3.2)$$

and

$$\{x, \ell(A)\} > \{\ell(A)\}. \quad (3.3)$$

By WIND we have, from (3.2) and (3.3), respectively,

$$D - \{x\} \geq D \quad (3.4)$$

and

$$D \geq D - \{x\}. \quad (3.5)$$

(3.1) follows from (3.4) and (3.5).

Let $A = \{g(A), a_1, a_2, \dots, a_t, \ell(A)\}$. Then by repeated application of (3.1),

$$\begin{aligned} & [A \sim A - \{a_1\}] \& [A - \{a_1\} \sim A - \{a_1, a_2\}] \\ & \& \dots \& [A - \{a_1, \dots, a_{t-1}\} \sim \{g(A), \ell(A)\}]. \end{aligned}$$

Hence by the fact that \geq is an ordering, $A \sim \{g(A), \ell(A)\}$. ■

Remark 3.1. The counterpart of Theorem 3.3 where R is an ordering but not necessarily a linear ordering can be easily proved. A will turn out to be "indifferent" under \geq to a set consisting of all R -greatest and R -least elements in A .

Remark 3.2. In view of the negative results, Theorem 3.1 and 3.2, it is worth noting that there exists an ordering \geq over χ satisfying WDP and WIND. An example is given by Barbera and Pattanaik [3].

Our next result characterizes the set of orderings \succcurlyeq over χ which satisfy WDP and WIND with respect to R . This is done by suitably specifying a class of real-valued functions defined over

$$Z \equiv \{(i, j) \mid i, j \in N_\chi \text{ and } i \succcurlyeq j\}.$$

THEOREM 3.4. (3.4.1) *Let \succcurlyeq be an ordering over χ satisfying WDP and WIND with respect to R . Then there exists a function $f: Z \rightarrow \mathbb{R}$ such that for all $A, B \in \chi$,*

$$A \succcurlyeq B \quad \text{iff } f(\max N_A, \min N_A) \geq f(\max N_B, \min N_B). \quad (3.6)$$

For every such function f , for all $t, t' \in N_\chi$, and for all $(g, h), (i, j) \in Z$,

$$f(t, t) > f(t, t') > f(t', t') \quad \text{if } t > t'; \quad (3.7)$$

$$f(g, h) > f(i, j) \rightarrow f(\max(t, g), h) \geq f(t, j) \quad \text{if } g \neq t > i, \quad (3.8)$$

and

$$f(g, h) > f(i, j) \rightarrow f(g, t) \geq f(i, \min(t, j)) \quad \text{if } h > t \neq j. \quad (3.9)$$

(3.4.2) *Conversely, let $f: Z \rightarrow \mathbb{R}$ be a function satisfying (3.7), (3.8), and (3.9). Then the binary relation \succcurlyeq over χ , induced by f by the rule (3.6), is an ordering and satisfies WDP and WIND with respect to R .*

Proof. (3.4.1) Let \succcurlyeq be an ordering over χ satisfying the specified properties. Let $f: Z \rightarrow \mathbb{R}$ be such that for all $(r, s), (t, u) \in Z, f(r, s) \geq f(t, u)$ iff $\{x_r, x_s\} \succcurlyeq \{x_t, x_u\}$ (since Z is finite, such a function clearly exists). Since \succcurlyeq satisfies WDP and WIND, by Theorem 3.3, for all $A \in \chi, A \sim \{g(A), \ell(A)\}$. Hence for all $A, B \in \chi$,

$$A \succcurlyeq B \quad \text{iff } \{g(A), \ell(A)\} \succcurlyeq \{g(B), \ell(B)\},$$

from which (3.6) follows immediately given the definition of f .

Consider a function $\hat{f}: Z \rightarrow \mathbb{R}$ which satisfies (3.6). Since \succcurlyeq satisfies WDP, \hat{f} clearly satisfies (3.7). Let $(g, h), (i, j) \in Z$ be such that $\hat{f}(g, h) > \hat{f}(i, j)$ and let $t \in N_\chi$. Let $m = \max(t, g)$. Since $\hat{f}(g, h) > \hat{f}(i, j)$, by (3.6),

$$\{x_g, x_h\} > \{x_i, x_j\}. \quad (3.10)$$

Suppose $g \neq t > i$. There are two possibilities: either $t > h$ or $t \leq h$. Suppose $t > h$. Then by Theorem 3.3,

$$\{x_t, x_g, x_h\} \sim \{x_m, x_h\} \quad \text{and} \quad \{x_t, x_i, x_j\} \sim \{x_i, x_j\}. \quad (3.11)$$

Since $g \neq t > i \geq j$ and $t > h$, t is distinct from g, h, i and j . Hence by (3.10) and WIND, $\{x_t, x_g, x_h\} \succcurlyeq \{x_t, x_i, x_j\}$. Hence, noting (3.11), it follows that

$\hat{f}(\max(t, g), h) \geq \hat{f}(t, j)$ for the case where $t > h$. Now suppose $t \leq h$. Then given $m = g \geq h \geq t > i \geq j$, by WDP and transitivity of \geq , we have $\{x_m, x_h\} \geq \{x_t\} \geq \{x_i, x_j\}$, and hence $\hat{f}(\max(t, g), h) > \hat{f}(t, j)$. This completes the proof of the fact that \hat{f} satisfies (3.8). That \hat{f} satisfies (3.9) can be shown similarly.

(3.4.2) Consider a function f satisfying (3.7), (3.8), and (3.9). Let \geq be induced by f through (3.6). It is obvious that \geq is an ordering. WDP follows from (3.7). We have to show that \geq satisfies WIND.

First, we show that for all $(g, h), (i, j) \in Z$,

$$f(g, h) \geq f(i, j) \quad \text{if } g \geq i \quad \text{and} \quad h \geq j. \quad (3.12)$$

Equation (3.12) is obvious if $g = i$ and $h = j$. Without loss of generality assume $h > j$. Then by (3.7), $f(h, h) > f(h, j)$ and hence noting (3.8), $f(g, h) \geq f(g, j)$. If $g = i$, (3.12) follows immediately. Suppose $g > i$. Then $f(g, i) > f(i, i)$ by (3.7). Then noting (3.9), $f(g, j) \geq f(i, j)$. Given $f(g, h) \geq f(g, j)$, (3.12) follows immediately. Thus (3.12) holds in all cases.

Now let $A, B, C \in \mathcal{X}$ be such that $A > B$ and $A \cap C = B \cap C = \emptyset$. Since $A > B$,

$$f(\max N_A, \min N_A) > f(\max N_B, \min N_B). \quad (3.13)$$

By (3.12), it follows that not $[\max N_B \geq \max N_A \ \& \ \min N_B \geq \min N_A]$. Then at least one of (3.14), (3.15), and (3.16) holds:

$$\max N_A \geq \max N_B \ \& \ \min N_A \geq \min N_B; \quad (3.14)$$

$$\max N_A > \max N_B \ \& \ \min N_A < \min N_B; \quad (3.15)$$

$$\max N_A < \max N_B \ \& \ \min N_A > \min N_B. \quad (3.16)$$

If (3.14) holds, then clearly $\max N_{A \cup C} \geq \max N_{B \cup C}$ and $\min N_{A \cup C} \geq \min N_{B \cup C}$. Then by (3.12) and the definition of \geq , it follows that $A \cup C \geq B \cup C$.

Suppose (3.15) holds. Then

$$\max N_{A \cup C} \geq \max N_{B \cup C} \quad (3.17)$$

and

$$\min N_{B \cup C} \geq \min N_{A \cup C}. \quad (3.18)$$

If $\min N_{B \cup C} = \min N_{A \cup C}$, then given (3.17), by (3.12) and the definition of \geq , we would have $A \cup C \geq B \cup C$. Suppose $\min N_{B \cup C} > \min N_{A \cup C}$. Then given that $\min N_B > \min N_A$ and $A \cap C = B \cap C = \emptyset$, it is clear that

$$\min N_B \geq \min N_{B \cup C} > \min N_A = \min N_{A \cup C}. \quad (3.19)$$

Then

$$\begin{aligned} f(\max N_A, \min N_{A \cup C}) &= f(\max N_A, \min N_A) \\ &> f(\max N_B, \min N_B) && \text{by (3.13) (3.20)} \\ &\geq f(\max N_B, \min N_{B \cup C}) && \text{by (3.12).} \end{aligned}$$

Clearly, $\max N_{A \cup C} \geq \max N_A$. If $\max N_{A \cup C} > \max N_A$, then

$$\begin{aligned} f(\max N_{A \cup C}, \min N_{A \cup C}) &\geq f(\max N_{A \cup C}, \min N_{B \cup C}) && \text{by (3.8 and (3.20)} \\ &\geq f(\max N_{B \cup C}, \min N_{B \cup C}) && \text{by (3.17) and (3.12).} \end{aligned}$$

If $\max N_{A \cup C} = \max N_A$, then given $A \cap C = \emptyset$ we have $\max N_A > \max N_C$, and hence noting (3.15), we have $\max N_{A \cup C} = \max N_A > \max N_{B \cup C}$. Then by (3.20) and (3.8), $f(\max N_{A \cup C}, \min N_{A \cup C}) \geq f(\max N_{B \cup C}, \min N_{B \cup C})$. Thus in all cases if (3.15) holds, then $f(\max N_{A \cup C}, \min N_{A \cup C}) \geq f(\max N_{B \cup C}, \min N_{B \cup C})$, and hence $A \cup C \succcurlyeq B \cup C$.

Similarly it can be shown that if (3.16) holds, then $A \cup C \succcurlyeq B \cup C$.

This completes the proof. ■

IV. AN APPLICATION IN WELFARE ECONOMICS

Our preceding discussion has some interesting implications for welfare economics where social welfare judgements of an individual have often been interpreted in terms of self-interested choices which the individual would make in certain specified situations of risk (see Hařsanyi [6] and Vickrey [15]) or uncertainty (see Rawls [13]). Let $\{1, \dots, t\}$ be the set of individuals constituting the society and let S be the set of all possible alternative social states. Let every social state $x \in S$ be characterized by $(a_{x1}, b_{x1}), \dots, (a_{xt}, b_{xt})$, where a_{xi} refers to the subjective features of individual i and b_{xi} refers to i 's objective position in the social state x . Let $X = \{(a_{xi}, b_{xi}) \mid x \in S \text{ and } i \in \{1, \dots, t\}\}$. Let k be any individual and let R^k be his ordering over X . It is assumed that R^k reflects ordinal comparisons of the *levels* of welfare of different individuals in alternative social states. In the Rawlsian spirit, k 's ranking of any two social states x and y , in terms of social welfare, is interpreted to be his ranking of the uncertain prospects $\{(a_{x1}, b_{x1}), \dots, (a_{xt}, b_{xt})\}$ and $\{(a_{y1}, b_{y1}), \dots, (a_{yt}, b_{yt})\}$. Intuitively, k 's ranking of x and y in terms of social welfare is thus interpreted to be his egoistic ranking of the social states when having made his choice between x and y , k has to acquire the subjective features as well as the objective position of some individual $g \in \{1, \dots, t\}$ in the chosen social state, but is uncertain, at the time of choosing between x and y , which individual g would be. It is

clear that the problem can be analyzed in terms of our framework of inducing an ordering over $2^X - \{\emptyset\}$, given the ordering R^k over X . If the egoistic behaviour of the individual under uncertainty is such that \succsim will satisfy WDP and WIND, then by our Theorem 3.3 and Remark 3.1 it follows that k 's ranking of any two social states, x and y , in terms of social welfare will depend solely on the R^k -greatest and R^k -least elements in $\{(a_{x1}, b_{x1}), \dots, (a_{xt}, b_{xt})\}$ and $\{(a_{y1}, b_{y1}), \dots, (a_{yt}, b_{yt})\}$. That is, k 's social welfare ranking of x and y will depend solely on the welfare levels of the best-off and worst-off individuals in each of the two social states.

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